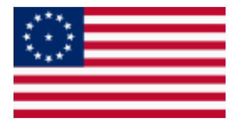
Problem of the Week

September 24, 2011

Flags of the Future



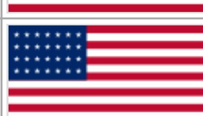


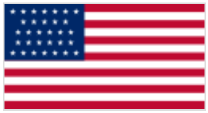
The U.S. flag has exactly one star per state, so the flag has changed a lot over time. More often than you might realize…when I was born, for example, there were only 48 states (yes, I really am that old). Even if the number of stars is set, it isn't clear how to arrange them. Since 1912 the pattern has been set officially but it isn't clear how that arrangement is determined. At various times, different sorts of patterns have been used and we can look at the historical flags to determine likely future ones. Above, you see some of the officially recognized flags back when we only had 13 states. I have attached a few pages showing the most commonly used historical flags (who knew there were pentagrams?). As you will see, some numbers don't work with some patterns.

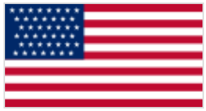
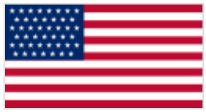
Question 1: Can you describe each of the patterns employed?

Question 2: Suppose we had 51 states. Which patterns might be used? (That is, which of the historically used patterns can be used with 51 stars?).

Question 3: What about 52? 53? And so on… Are any numbers especially hard to picture? Historically, for example, adding Oregon seems to have been a bit hard on the flag makers.

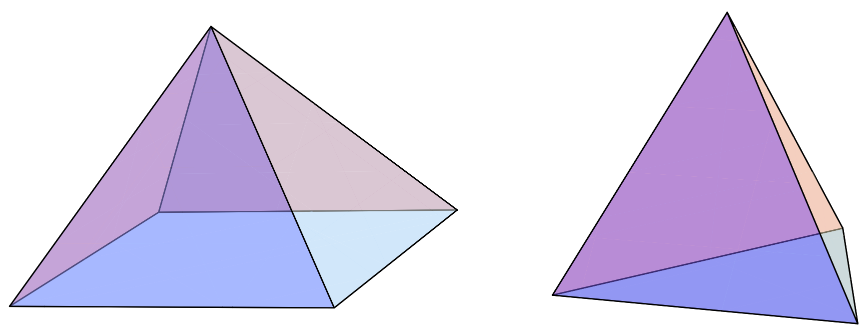
Stars	Stripes	Image	New States	Period	
15	15		Vermont, Kentucky	May 1, 1795 – July 3, 1818	23 years (278 months)
20	13	 	Indiana, Louisiana, Mississippi, Ohio, Tennessee	July 4, 1818 – July 3, 1819	1 year (12 months)
21	13		Illinois	July 4, 1819 – July 3, 1820	1 year (12 months)
23	13		Alabama, Maine	July 4, 1820 – July 3, 1822	2 years (24 months)
24	13		Missouri	July 4, 1822 – July 3, 1836 1831 term "Old Glory" coined)	14 years (168 months)
25	13		Arkansas	July 4, 1836 – July 3, 1837	1 year (12 months)
26	13	 	Michigan	July 4, 1837 – July 3, 1845	8 years (96 months)
27	13		Florida	July 4, 1845 – July 3, 1846	1 year (12 months)
28	13		Texas	July 4, 1846 – July 3, 1847	1 year (12 months)
29	13	 	Iowa	July 4, 1847 – July 3, 1848	1 year (12 months)

30	13		Wisconsin	July 4, 1848 – July 3, 1851	3 years (36 months)
31	13		California	July 4, 1851 – July 3, 1858	7 years (84 months)
32	13		Minnesota	July 4, 1858 – July 3, 1859	1 year (12 months)
33	13	   	Oregon	July 4, 1859 – July 3, 1861	2 years (24 months)
34	13		Kansas	July 4, 1861 – July 3, 1863	2 years (24 months)
35	13		West Virginia	July 4, 1863 – July 3, 1865	2 years (24 months)
36	13	 	Nevada	July 4, 1865 – July 3, 1867	2 years (24 months)
37	13	 	Nebraska	July 4, 1867 – July 3, 1877	10 years (120 months)
38	13	 	Colorado	July 4, 1877 – July 3, 1890	13 years (156 months)

43	13		Idaho, Montana, North Dakota, South Dakota, Washington	July 4, 1890 – July 3, 1891	1 year (12 months)
44	13		Wyoming	July 4, 1891 – July 3, 1896	5 years (60 months)
45	13		Utah	July 4, 1896 – July 3, 1908	12 years (144 months)
46	13		Oklahoma	July 4, 1908 – July 3, 1912	4 years (48 months)
48	13		Arizona, New Mexico	July 4, 1912 – July 3, 1959	47 years (564 months)
49	13		Alaska	July 4, 1959 – July 3, 1960	1 year (12 months)
50	13		Hawaii	July 4, 1960 – present	50 years (602 months)

Problem of the Week   October 12, 2010

An Unfortunate PSAT Question



Shown above are two solids:

- a regular pyramid, which has a square base and four equilateral triangles on the sides
- a regular tetrahedron which comprises four equilateral triangles.

Between them, they have 9 faces. Supposing that we have made all the side lengths the same, we can glue them together on one triangular face. How many faces has the new object got?

Note 1: I am told that this question appeared on a PSAT exam a few years ago and that the expected answer was the obvious one (7). Distressingly, this obvious answer is not correct.

Note 2: Personally, I find it almost impossible to visualize this. Might be best to try to make the two solids.

Problem of the Week

October 20, 2010

Baseball Odds

Sports commentators often talk about probability and odds, but it isn't easy to separate good arguments from bad; anecdotal (and sentimental) reasoning often takes the place of solid math. For example, on a recent broadcast the announcer confidently asserted that *Momentum* was critical. He meant that the winner of a game had increased odds of winning the next. To prove it, he asserted that the winner of the first game in a 7 game playoff series historically won 58% of the time. (I don't have the data on all the playoffs, but I checked the World Series and got 61%.)

Note: to be clear, the winner of a seven game series is the first to win four games.

Question 1: Assuming that each game is just a 50-50 coin toss, what are the odds that the winner of the first game will win the series? Is the 58% persuasive evidence of momentum? Is 61% ?

Question 2: (Harder) Let's model momentum by saying that the winner of a game has a 60% chance of winning the next. Now, what are the odds that the winner of the first game in a seven game series will win the series? (Bear in mind that if you lose a game, your opponent will acquire the momentum!)

Question 3: (Open Ended) can you come up with any sensible way to explain the 61%?

Problem of the Week

October 20, 2010

Baseball Odds (easier version)

It's hard to work with seven game series, so take smaller ones.

1. assuming every game is a 50-50 game toss, what are the odds that the winner of the first game in a three game series will win the series? (if this is too easy, try a five game series).

2. Now try to add momentutm. Assume that the winner of the most recent game now has a 2/3 chance of winning the next. Now what are the odds that the winner of the first game in a three game series will win the entire series? (again, if this is too easy, try a five game series).

Here is an easy way to play this game for real: take an ordinary die. Each number, from 1…6 comes up with equal odds, so if A bets on {1,2,3} A will win with 50-50 odds. To shift the odds, suppose A bets on {1,2,3,4}. Now, A will win with 2/3 odds (why?).

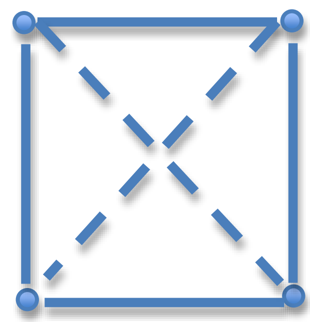
To play the momentum game, just suppose that, if A wins the first game, then A gets to bet on {1,2,3,4}. Likewise, if A then loses the second game, B will get to bet on {1,2,3,4} for the third.

This gives an easy way to check your answers! Just play the game over and over again and keep track of the winner. Of course, this works just as well for the seven game series.

Problem of the Week

November 2, 2010

Four Points



Setting four points as the corners of a square gives you exactly two lengths: the length of a side (solid), and the length of a diagonal (dashed). There are five other ways to arrange four points in the plane that have this property; can you find them?

To rephrase: In general, any arrangement of four points in the plane defines six line segments (check this!). This problem requires that there be only two lengths among these six segments.

Note 1: it is impossible to do this with only one length (why is this?)

Note 2: it seems that almost no one can come up with all of the remaining five pictures; though people seem to miss each of the possibilities with roughly equal odds.

Problem of the Week

November 15, 2010

An Odd Game

Pick a number. It can be as long as you like (though you will have to do a little arithmetic on it). Let's say you pick

N = 29,399,182,084,972

Step 1: re-order the digits in your number from least to greatest. Throw out any zeroes (they don't change the answer). With the given value you'd now get

1,222,347,889,999

Step 2: look at the rightmost block, the one consisting of the highest digit in your number (in our example this block consists of four 9's). Discard all but one of these. With our example, you'd now get

1,222,347,889

Step 3: Multiply this value by 9 (this is the only messy stage!). In our example you'd get

11,001,131,001

Step 4: Add up the digits in this value. In our example you get 9.

Try this with a bunch of examples, some short and some long. What do you get?

Why on earth does this happen? (note: this isn't exactly hard, but it certainly is puzzling. See if you can work it out for all 3 digit numbers, or all 4 digit numbers).

Problem of the Week

December 1, 2010

What's the first number in the dictionary?

To be clear, we're only talking about whole numbers here. To get a whole number into the dictionary, we just write it out in the usual way, ignoring spaces and hyphens and omitting any "ands". Thus, the number 16, 683, 986 would be written out as sixteen million six hundred eighty three thousand nine hundred eighty six.

If that question is too easy: What is the first *odd* number in the dictionary?

Hint: that one is somewhat larger.

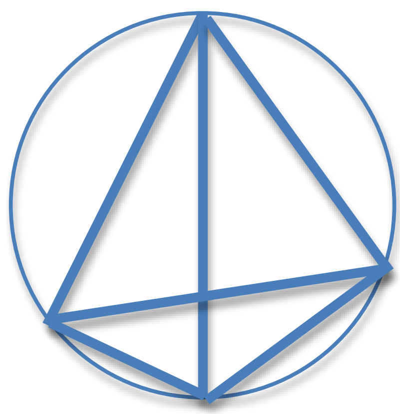
For those who can use a computer: What is the first *prime* number in the dictionary?

Problem of the Week

January 4, 2011

Bad Pattern

Pick a positive number n, let's say 4. Draw a circle and put n points on it randomly (not in any neat arrangement). Now connect the points by lines in all possible ways. Having chosen 4, our picture looks like this:



How many regions in the circle have we carved out? Well, we can see from the picture that there are 8 in our example.

Note: if necessary, move your points a little to make sure that no three of the lines meet at a point inside the circle.

Problem: how many regions do you get for n = 1, 2, or 3 ? What do you think the pattern is?

Harder Problem: How many regions do you get if n = 5?

Much harder Problem: How many do you get if n = 6?

Really much, much harder problem: What is the correct pattern?

Problem of the Week

February 9, 2011

Poor Man's Fibonacci Numbers

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …

To get the Fibonacci numbers: start with 1, 1. Then, to get the next number in the sequence, add the two that precede it. Thus the third entry should be 1 + 1 = 2. The fourth entry is then 2 + 1 = 3, and so on.

Here's a simple question: Just take the last digits of each of the Fibonacci numbers. As you can see from the list above, this new series goes: 1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, …. Does this series ever start to repeat itself? When?

Note: you can certainly attack this by just writing it out. Even so, you have to be careful how you do it! The numbers get very large, very quickly. It is also possible to decide whether or not it repeats without actually computing much; more abstract, but a lot less work!

Solution(Fibonacci).

In general: yes, the sequence must always start to repeat. There are only finitely many pairs of possible remainders so eventually we must get the same pair twice.

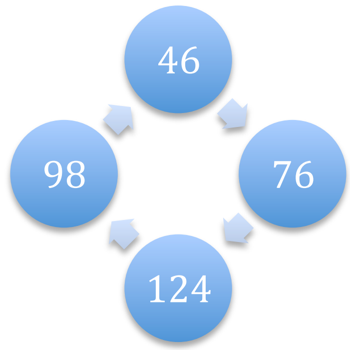
Indeed, 1, 1 must always recur. This is much less obvious. To see it, suppose that (x, y) is the first pair that repeats. We know from the first paragraph that some pair repeats, (x, y) is just meant to be the first one to get repeated. If (x, y) is (1, 1) we are done. If not, then find the first occurrence of (x, y) in the sequence. Let (a, b) be the pair which precedes (x, y) [if the pair (x, y) is the pair (1, 2) then we say that the preceding pair is (0, 1)]. Now look at the second time (x, y) occurs and let (c, d) be the pair which precedes it. We claim that (c, d) = (a, b). To see this, note that the pair which follows (a, b) is (a + b, a + 2b) so our assumption implies that (a + b, a + 2b) = (c + d, c + 2d) which quickly implies that a = c and b = d (Why??). But then the pair (a, b) also recurs! And this pair comes before (x, y), so we have a contradiction.

The case of last digits is most easily done by hand. How long might we have to search? Well…let's count pairs. There are 10*9 = 90 pairs in which the numbers are distinct, and 10 in which they coincide. Hence there are 100 pairs all told, so in principle we might have to search 200 terms of the sequence! Annoying, but perfectly possible. In the end, we aren't as unlucky as all that. The 61^{st} and 62^{nd} terms are 1, 1.

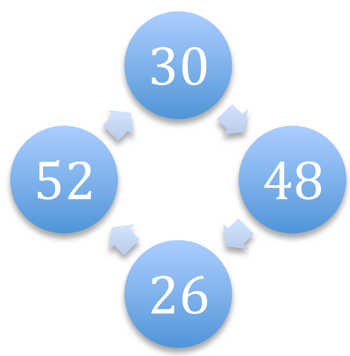
Note: to make your life easier, don't compute all the numbers out! As we only care about the last digit, just keep single digits. Thus, 3 + 5 is 8 as always, but 3 + 9 is 2! Working this way makes the arithmetic a lot easier.

Problem of the Week   March 1, 2011

Circle Game



Pick any 4 numbers and arrange them in a circle, as shown. Now, replace each number with the difference between it and the number one clockwise away from it. In each case, you take the larger less the smaller (so the numbers stay non-negative). Starting with the numbers shown, as an example, we would get:

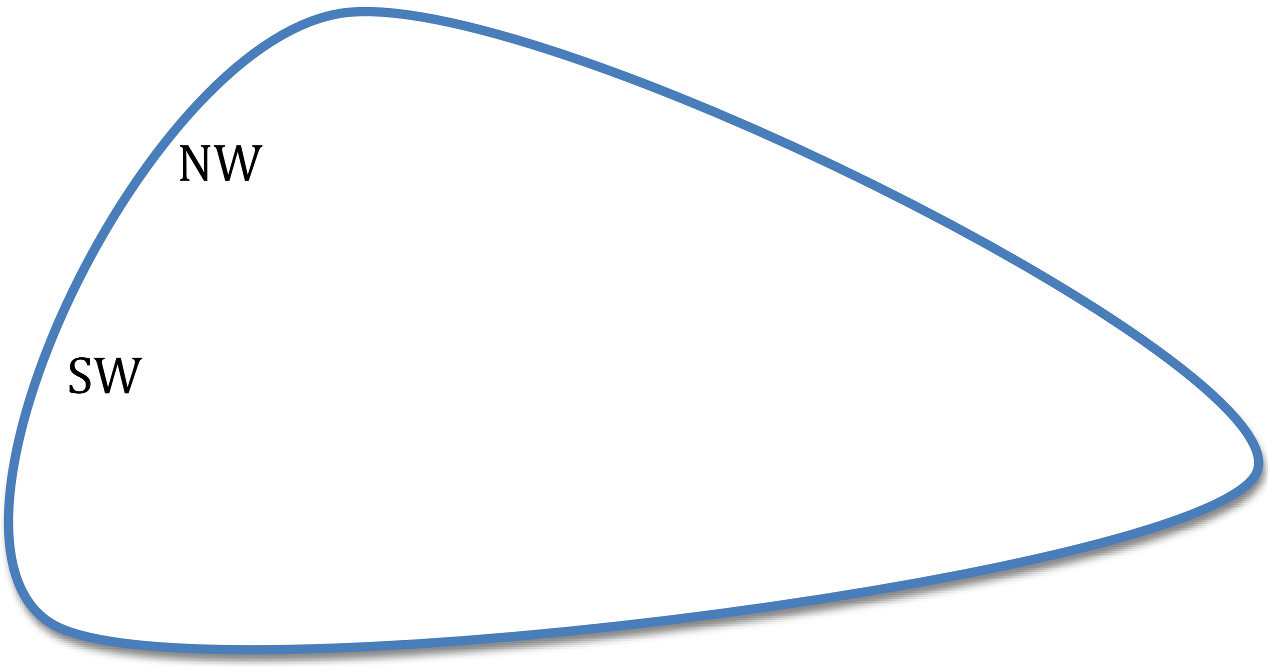


Keep on doing this. What happens? Try different starting numbers…does the same thing keep happening? Can you come up with a reason for this behavior? Does the same thing happen if you started with three numbers? With five numbers?

Note: This game was played by prisoners of war during World War II who passed the time by seeking starting points which could last as long as possible. How long can you keep it going?

Problem of the Week   March 29, 2011

Treasure Map



You are on a tropical island, in possession of a treasure map which, sadly, is in poor condition. You can't read all of it, but you can make out the following:

…Four markers there be, in a rectangle. The treasure is buried 27 yards from the NorthWest corner, 18 yards from the NorthEast, 38 yards from the SouthEast, and from the SouthWest…..

That's all you can read. The distance from SW is not legible. Worse, you can find the NW and SW markers but the other two have crumbled to dust.

Where is the treasure?

Problem of the Week

April 26, 2011

Greed

Reluctantly yielding to overwhelming public pressure, the head of a large corporation agrees to an unusual salary arrangement: every one of the 1,000 employees, including the head, receives the same salary (which, to keep it simple, let us say is 1 gold coin a week). The 1,000 total pay is fixed but the distribution of salaries may change by majority vote. At the end of each week, the head may propose a new distribution of the total 1,000 coins. Not a problem, says he, as no new arrangement will ever be implemented unless a majority of those who vote support it and, just to keep things fair, the head won't even get a vote. To keep it clear, let us imagine that any employee whose salary would increase under the proposed scheme votes for it and that any employee whose salary would decrease votes against it. Those whose salary would be unchanged don't vote at all (as was said, the head never votes). Sounds fair, right?

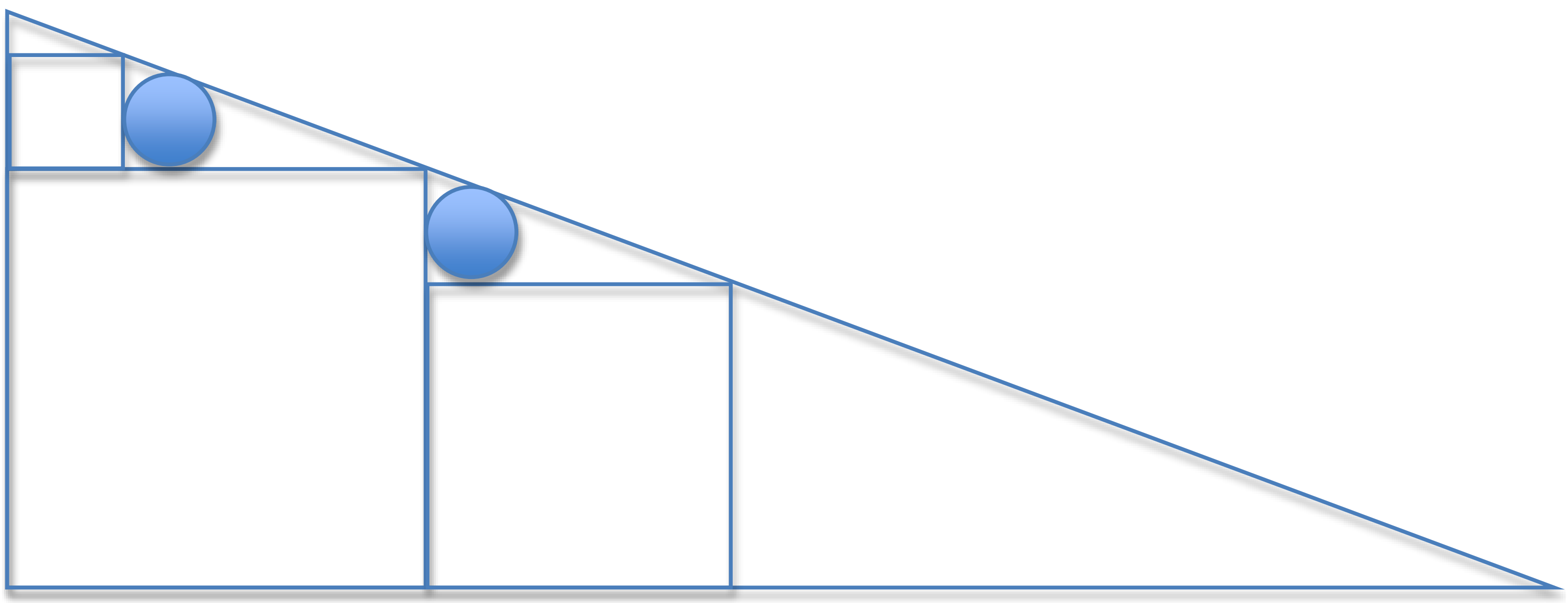
How big a salary can the head arrange for himself and how many weeks does it take him to get there?

Note: at first glance it may appear that nothing can ever change under these rules. After all, if the head proposes to give X's pay to Y, then Y votes yes, X votes no, and nothing happens. That's what makes this plan so convincing. What can the head do?

Problem of the Week

May 10, 2011

Triangles, Squares, and Circles



Various squares and circles are arranged in a right triangle, as shown. Are the two shaded circles equal?

Despite the imperfections of the drawing, you should suppose that any objects which appear to be touching actually are.

Note: it is perfectly possible to compute all the measurements in question in terms of the sides of the original triangle, but it is not necessary. The problem can be solved with no computation at all.